On Bell's Theorem

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Abstract

Bell's Theorem implies that quantum correlation of entangled particles as calculated in quantum mechanics violates elementary probabilistic inequalities. It is shown that the reason is a problem in scaling of detector directions.

Bell's Theorem [1] considers measurements of spin for entangled particles. The entangled wave function studied in this teorem is

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|\psi_{z+}\rangle \otimes |\psi_{z-}\rangle - |\psi_{z-}\rangle \otimes |\psi_{z+}\rangle)$$

where $|\psi_{j+}\rangle$ and $|\psi_{j-}\rangle$, $j \in \{x, y, z\}$, are the eigenvectors of the Pauli matrices σ_j corresponding to the eigenvalues 1 and -1 respectively. The spin of the first particle is detected with two detector directions a and a' and the spin of the second particle is detected in the second measurement with two detector parameters b and b'. Measurement of the spin means applying operators

$$(\sigma \cdot a) \otimes Id \quad , \qquad Id \otimes (\sigma \cdot b).$$

In both cases the spin measurement can only give the results spin up or spin down and collapses the wave function to direct products of eigenvectors of Pauli matrices. The quantum correlation

$$C(a,b)_q = \langle \phi | Id \otimes (\sigma \cdot b)(\sigma \cdot a) \otimes Id | \phi \rangle$$

gives the expected value for the empirical correlation

$$C(a,b)_e = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N_{++} + N_{--} + N_{+-} + N_{-+}}$$

where $N_{\alpha,\beta}$, $\alpha, \beta \in \{+,-\}$, is the number of cases when the first particle is measured spin α and the second spin β. A direct calculation shows that

$$C(a,b)_q = -b \cdot a.$$

The measurements of entangled pairs in the directions $a, b,\ a, b',\ a', b$ and a', b' form four time series that have only values ± 1. We can define binary probability variables A, B, A' and B' and assign to them probabilities of being 1 or -1 from these time series. The corretion between these variables is then the expectation value of the product of the variables, thus

$$C(a,b)_p = E(AB).$$

Binary variables satisfy certain inequalities, called Bell's inequalities. The CHSH inequality is a convenient Bell's inequality for proving Bell's theorem:

$$C(a,b)_p + C(a,b')_p + C(a',b)_p - C(a',b')_p \leq 2. \tag{1}$$

Assiging detector directions as

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad a' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad b' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{2}$$

gives

$$C(a,b)_q + C(a,b')_q + C(a',b)_q - C(a',b')_q = 2\sqrt{2} > 2 \tag{3}$$

showing that $C(a,b)_q \neq C(a,b)_p$. Bell's inequality violations have been observed in experiments. However, the reason is quite simple. The detector values are normalized to give

$$a \cdot b = a' \cdot a' = b \cdot b = b' \cdot b' = 1 \tag{4}$$

which may initially seem correct, but it is not. The correct scaling is

$$\sum_i |a_i| = \sum_i |a'_i| = \sum_i |b_i| = \sum_i |b'_i| = 1. \tag{5}$$

Scaling the directions according to (5) removes the Bell's inequality violation in (2).

In order to show that the scaling (4) is incorrect, let the directions be scaled as in (4) and let us consider the first particle

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_{z+}\rangle - |\psi_{z-})$$

without focusing on the entanglement. It is measured by applying the operator $\sigma \cdot a$. The wave function $(\sigma \cdot a)|\psi\rangle$ collapses to one of the eigenvectors $|\psi_{m\alpha}\rangle$, $m \in \{x, y, z\}$, $\alpha \in \{+, -\}$, of Pauli matrices and the corresponding eigenvalue is α. If the first particle collapses to $|\psi_{m\alpha}\rangle$ the second particle collapses to $|\psi_{m\beta}\rangle$, $\beta \neq \alpha$. In the second measurement this collapsed second particle collapses to one of the eigenvectors $|\psi_{n\gamma}\rangle$, $n \in \{x, y, z\}$, $\gamma \in \{+, -\}$. Because $|\langle\psi_{n\gamma}|\psi_{m\beta}\rangle|^2 = \frac{1}{2}$ if $n \neq m$, the probability of $|\psi_{m\alpha}\rangle$ collapsing to $|\psi_{n+}\rangle$ is the same as the probability that it collapses to $|\psi_{n-}\rangle$. As the eigenvalues for $|\psi_{n+}\rangle$ and $|\psi_{n-}\rangle$ are opposites, these contributions to the correlation of the first and the second particle cancel. There remains the collapse of $|\psi_{m\beta}\rangle$ to $|\psi_{m\gamma}\rangle$, $\gamma \in \{+, -\}$. As $|\langle\psi_{m\gamma}|\psi_{m\beta}\rangle|^2 = \delta_{\beta=\gamma}$ if can only collapse to $|\psi_{m\beta}\rangle$.

In the scaling (4) the wave function $|\psi\rangle$ of the first particle collapses to $|\psi_{m\alpha}\rangle$ with the probability a_m. Because of entanglement, the second particle collapses after the first measurement to $|\psi_{m\beta}\rangle$, $\beta \neq \alpha$, with the same probability a_m. In the second measurement this collapsed second particle collapses to either $|\psi_{n+}\rangle$ or $|\psi_{n-}\rangle$ with the weight $|b_n|$. The sum of these weights must be one. Thus, the probability of the collapse is

$$\frac{b_n}{\sum_i |b_i|.}$$

Thus, the quantum correlation is not $-b \cdot a$. It is

$$C(a, b)_q = \frac{-b \cdot a}{\sum_i |b_i|}. \tag{8}$$

Correcting the quantum correlation removes Bell's inequality violation in (2).

However, this is not the logical way to correct it. We certainly want that

$$\langle\psi|(\sigma \cdot a)(\sigma \cdot a)|\psi\rangle = a \cdot a.$$

This holds if we scale as in (5):

$$\sum_i |a_i| = 1.$$

Adopting this scaling we see that

$$\sum_i a_i^2 < 1 \tag{9}$$

and may wonder where the missing probability is. It is in the contributions that cancelled in the calculation of quantum correlation and which also cancel in the calculation of autocorrelation (8). Indeed, the probability of $|\psi\rangle$ collapsing to either $|\psi_{m+}\rangle$ or $|\psi_{m-}\rangle$ is not a_m^2. That is only the part that is seen. Calculating autocorrelation (8) can be understood as two measurements, as with the correlation of two particles. The first measurement collapses $|\psi\rangle$ to eigenvectors $|\psi_{m+}\rangle$ or $|\psi_{m-}\rangle$ with the probability $|a_m|$ in the scaling (5). In the second measurement these eigenvectors are further collapsed to $|\psi_{n\gamma}\rangle$ and the probability of collapses to either $|\psi_{n+}\rangle$ or $|\psi_{n-}\rangle$ is $|a_n|$. Thus, the sum of the probabilities of eigenvectors to which $|\psi_{m+}\rangle$ or $|\psi_{m-}\rangle$ collapse is

$$|a_m|(|a_x|+|a_y|+|a_z|)=|a_m|$$

of which we see only the part $|a_m|^2$ as the other parts cancel in the measurement. The total probability of scaling (5) is one, though it appears, as in (9), that probabilities do not sum to one. This phenomenon explains why experiments have verified violations of Bell's inequality. In these experiments the probabilities have been derived from the numbers of detected particles and their sum has been scaled to 1 as in (9). This ignores the probability of contributions that cancel.

Bell's Theorem is sometimes explained by the following example. Assume that in a test of entangled particles the detectors in both sides are perfectly aligned, $a_z = b_z = 1$. We see perfect anticorrelation. Then move the detector of the first particle in the (x,z)-plane to a small angle $\alpha = \gamma/2$ so that there are 1% errors in detection. Moving the detector of the second particle in the (x,y)-plane to the angle $\beta = -\gamma/2$ must also introduce 1% errors. Thus, there should be 2% errors, but according to quantum mechanics, and experiments, there will be 4% errors. The number of errors was $\sin^2(\gamma/2)$ when the (small) angle between the detectors was $\gamma/2$ and it grows to

$$\sin^2(\gamma) = (2\sin(\gamma/2)\cos(\gamma/2)^2 \approx 4\sin^2(\gamma/2)),$$

to four times as large when the angle between the detectors is γ.

In reality, there is no mystery. The number of errors is related to the x-coordinate of the detector as $\sin^2(\gamma/2) = a_x^2$ thus $a_x = \sin(\gamma/2)$. In a communication system analogy we can think that the fraction a_x of the bits have errors.

When β is set to $\gamma/2$ the errors in the communication analogy grow to about $2a_x$. The number of noticed errors is thus about $(2a_x)^2$, that is, four times as many errors as when the angle was only in one side.

The mathematical form of the quantum correlation $-b \cdot a$ can be expressed with the angle θ between the detectors as

$$-b \cdot a = -b_z a_z - b_x a_x = -|b||a| \cos(\alpha) \cos(\beta) - |b||a| \sin(\alpha) \sin(\beta)$$
$$= -\cos(\alpha - \beta) = -\cos(\theta)$$

as $|a| = |b| = 1$ by the norming (4), which is used when deriving this mathematical form of the correlation and also in experiments that have confirmed the form $-\cos(\theta)$.

The function $-\cos(\theta)$ is -1 if $\theta = 0$, zero if $\theta = \pi/2$ and 1 if $\theta = \pi$. Sometimes it is argued that as a classical correlation should be a linear function and the linear function fitting to these three points is $\frac{2}{\pi}\theta$, but as experiments confirm that $-\cos(\theta)$ is the correct form, this is a demonstration that quantum mechanics differs from classical physics.

Quantum mechanics certainly differs from classical physics, there is e.g. the collapse of wave functions, but this particular mathematical form of quantum correlation does not touch those issues. The correlation should indeed classically be a linear function, but a linear function of $|a_x|$. As $|a_x| = \sin(\theta)$ we have to look for a linear function agreeing on those three points. Two free parameters is not enough to fit the three points. We have to still take a linear shift of θ in the inside function $\sin(\theta)$. Thus, we look for a solution of the type

$$C(a,b)_q = k \sin(\theta + \gamma) + r$$

where k, r and γ are to be determined. From the three points we get three equations and the solution is $k = -1$, $r = 0$ and $\gamma = \frac{\pi}{2}$ giving the correct quantum correlation $C(a,b)_q = -\cos(\theta)$.

Hopefully this short note has helped to demystify Bell's Theorem. The reason for the failure of Bell's inequalities seems to be incorrect scaling of detector directions and it has nothing to do with hidden parameters in the EPR paradox.

References:

[1] John Bell,*On the Einstein Podolsky Rosen Paradox. Physics.* 1 (3):195-200, 1964.